Exc 1

answer model Test 1

1a $\|A\|_\infty = \max\limits_{1\le i\le 3} \sum_{j=1}^{3} |a_{ij}|$: maximum of absolute row sum

$\mathcal{K}(A) = \|A\|_\infty \|A^{-1}\|_\infty$

$\|A\|_\infty = \max(1+10^{-10}+10^{-5}, 1+10^{-10}+10^{-5}, 1) = 1+10^{-10}+10^{-5}$

$\|A^{-1}\|_\infty = \max(1, 1-10^{-10}+|-10^{-5}+10^{-15}|, 1+|-10^{-5}+10^{-15}| + |-10^{-20}|$

$= \max(1, 1-10^{-10}+10^{-5}-10^{-15}, 1+10^{-5}-10^{-15}+10^{-20})$

$= 1+10^{-5}-10^{-15}+10^{-20}$

$\Rightarrow \mathcal{K}(A) = (1+10^{-10}+10^{-5})(1+10^{-5}-10^{-15}+10^{-20})$

$= 1+2.10^{-5}+2.10^{-10}+10^{-30}$

Note: $\mathcal{K}(A) \simeq 1$

1b

$$\begin{array}{ccc:c} 10^{-10} & 10^{-5} & 1 & 1 \\ 10^{-5} & 1+10^{-10} & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \xrightarrow[3 - 10^{10}(1)]{2 - 10^{5}(1)} \begin{array}{ccc:c} 10^{-10} & 10^{-5} & 1 & 1 \\ 0 & 10^{-10} & -10^{5} & -10^{5} \\ 0 & -10^{5} & -10^{10} & 1-10^{10} \end{array}$$

$$\xrightarrow{3+10^{15}(2)} \begin{array}{ccc:c} 10^{-10} & 10^{-5} & 1 & 1 \\ 0 & 10^{-10} & -10^{5} & -10^{5} \\ 0 & 0 & -10^{10}-10^{20} & 1-10^{10}-10^{20} \end{array} \quad (*)$$

(*) due to machine accuracy $1-10^{10}-10^{20} = -10^{10}-10^{20}$

$\Rightarrow x_3 = \dfrac{-10^{10}-10^{20}}{-10^{10}-10^{20}} = 1$

$\Rightarrow x_2 = (-10^{5}+10^{5}x_3)/10^{-10} = 10^{15}(-1+x_3) = 0$

$\Rightarrow x_1 = (1-10^{-5}x_2 - x_3)/10^{-10} = 0.$

1c partial pivoting

interchange rows such that smallest element of first column is in the last row, and largest element of first column in first row

$$\begin{pmatrix} 1 & 0 & 0 \\ 10^{-5} & 1+10^{-10} & 0 \\ 10^{-10} & 10^{-5} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

rows 1 and 3 interchanged

$$\begin{matrix} (2) - 10^{-5}(1) \\ \Rightarrow \\ (3) - 10^{-10}(1) \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+10^{-10} & 0 \\ 0 & 10^{-5} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -10^{-5} \\ 1-10^{-10} \end{pmatrix}$$

$$\Rightarrow \quad (3) - \frac{10^{-5}}{1+10^{-10}}(2) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+10^{-10} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -10^{-5} \\ 1-10^{-10} + \frac{10^{-10}}{1+10^{-10}} \end{pmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = \frac{-10^{-5}}{1+10^{-10}}, \quad x_3 = 1-10^{-10} + \frac{10^{-10}}{1+10^{-10}}$$

$$x_3 = 1 + \frac{10^{-10} - 10^{-10} - 10^{-20}}{1+10^{-10}} = 1 - \underbrace{\frac{10^{-20}}{1+10^{-10}}}_{<u} = 1$$ in machine accuracy

1d solution with partial pivoting is correct

partial pivoting tries to reduce propagation of round off errors

Exc 2

2a 9×9 symmetric matrix

if node i not connected with node j:

$$a_{ij} = a_{ji} = 0$$

if node i connected with node j:

$$a_{ij} = a_{ji} = *$$

$$\begin{bmatrix} * & 0 & * & 0 & 0 & * & 0 & * & * \\ 0 & * & 0 & * & 0 & 0 & * & * & 0 \\ * & 0 & * & * & 0 & 0 & 0 & * & 0 \\ 0 & * & * & * & * & * & * & 0 & * \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & * & 0 & 0 & * \\ 0 & * & 0 & * & 0 & 0 & * & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & * & 0 & * & 0 & 0 & * \end{bmatrix} = A$$

2b Cuthill-McKee ordering

- start with node with least connections : 5
- next group; nodes connected to 5 : 4
- next group; nodes connected to 4 : $\underbrace{7}_{\text{degree 2}}, \overbrace{2, 3, 6, 9}^{\text{all degree 3, order arbitrary}}$
  order: nodes with lowest degree first in group
- next group; nodes connected to previous group : 8, 1
  first nodes connected to first node of previous group
  than nodes connected to second node of previous group
  etc. (only nodes that are not in previous groups)

Cuthill-McKee ordering {5, 4, 7, 2, 3, 6, 9, 8, 1}

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 5 & 4 & 7 & 2 & 3 & 6 & 9 & 8 & 1 \end{array} \\
\begin{array}{c} 5 \\ 4 \\ 7 \\ 2 \\ 3 \\ 6 \\ 9 \\ 8 \\ 1 \end{array} &
\begin{bmatrix}
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 \\
0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & * & 0 & 0 & 0 & * & 0 \\
0 & * & 0 & 0 & * & 0 & 0 & * & * \\
0 & * & 0 & 0 & 0 & * & * & 0 & * \\
0 & * & 0 & 0 & 0 & * & * & 0 & * \\
0 & 0 & 0 & * & * & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & *
\end{bmatrix}
\end{array} = A
$$

2c reverse Cuthill-McKee ordering : {1, 8, 9, 6, 3, 2, 7, 4, 5}
Cuthill-McKee ordering backwards

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 1 & 8 & 9 & 6 & 3 & 2 & 7 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 8 \\ 9 \\ 6 \\ 3 \\ 2 \\ 7 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
* & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & 0 & * & * & 0 & 0 & 0 & * & 0 \\
* & 0 & * & * & 0 & 0 & 0 & * & 0 \\
* & * & 0 & 0 & * & 0 & 0 & * & 0 \\
0 & * & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{bmatrix}
\end{array} = \tilde{A}
$$

non-zero's can occur in convex hull of $\tilde{A}$

$$
\tilde{A} \rightarrow L = \begin{bmatrix}
* & & & & & & & & \\
* & * & & & & & & & \\
* & * & * & & & & & & \\
* & * & * & * & & & & & \\
* & * & * & * & * & & & & \\
0 & * & * & * & * & * & & & \\
0 & 0 & 0 & 0 & 0 & * & * & & \\
0 & 0 & * & * & * & * & * & * & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{bmatrix}
$$

Exc 3

U unitair, orthonormal eigenvectors of AA^T
V unitair, orthonormal eigenvectors of A^TA

3a

A 2x3 matrix

regular SVD : $A = U\Sigma V^T$ $\quad$ U 2x2, Σ 2x3, V 3x3

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix} \quad V^T = \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}$$

$$\Rightarrow \Sigma V = \begin{pmatrix} \sigma_1 v_{11} & \sigma_1 v_{21} & \sigma_1 v_{31} \\ \sigma_2 v_{12} & \sigma_2 v_{22} & \sigma_2 v_{32} \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \end{pmatrix}}_{W^T}$$

$\Rightarrow A = UFW^T$ with U, F 2x2 $\quad$ U unitair

W 3x2 $\quad$ W first 2 columns of V $\Rightarrow W$ unitair

is SVD

3b

$\sigma_i = \sqrt{\lambda_i(A^TA)}$ or $\sigma_i = \sqrt{\lambda_i(AA^T)}$

note : A^TA and AA^T have the same non-zero eigenvalues

hence we can consider $\lambda_i(A^TA)$ or $\lambda_i(AA^T)$

since A^TA is 3x3 and AA^T is 2x2 we consider AA^T

- $AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda_1 = 2,\ u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  $\lambda_2 = 3,\ u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

  $\Rightarrow U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ F = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$ since U unitair, orthonormal eigenvectors of AA^T

- $A = UFW^T \Rightarrow A^T = WFU^T \Rightarrow W = A^T(U^T)^{-1}F^{-1}$

  $\Rightarrow W = A^TUF^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}$

3c

$$A^+ = WF^{-1}U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & -1/3 \\ 0 & 1/3 \end{bmatrix}$$

3d

pseudo-inverse ($x = A^+b$) gives solution shortest in 2-norm

general solution $Ax = b$: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$$\min_\alpha \left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\|^2 = \min_\alpha \left(\alpha^2 + (1-\alpha)^2 + (-2\alpha)^2 \right) = \min_\alpha (6\alpha^2 - 2\alpha + 1)$$

$\frac{d}{d\alpha}(6\alpha^2 - 2\alpha + 1) = 12\alpha - 2 = 0$ if $\alpha = \frac{1}{6}$ then norm minimal

$\Rightarrow$ pseudo-inverse solution : $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 5/6 \\ -1/3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$

note : you can check that A^+b with A^+ from 3c and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ gives indeed $x = \frac{1}{6}\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$